On the M–matrix inverse problem for symmetric singular tridiagonal matrices

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Abstract

A well–known property of an irreducible non–singular M–matrix is that its inverse is positive. However, when the matrix is an irreducible and singular M–matrix it is known that it has a generalized inverse which is non–negative, but this is not always true for any generalized inverse. We focus here in characterizing when the Moore–Penrose inverse of a symmetric, singular, irreducible and tridiagonal M–matrix is itself a M–matrix

Statement of the problem

Given $c_1,\dots,c_{n-1}>0$, the *Path*

and the *Schrödinger operator with potential* q

$$\mathsf{J}=\begin{bmatrix} d_1 & -c_1 & & & \\ -c_1 & d_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & d_{n-1} & -c_{n-1}\\ & & & -c_{n-1} & d_n\end{bmatrix},\ \text{where } d_i=q_i+c_i+c_{i-1}$$

- For which values $d_1,\dots,d_n$ is J a singular M–matrix?
- When $\mathsf{J}^\dagger$ is also an M–matrix?

When J is an M–matrix?

Given $c_1,\dots,c_n>0$ and $d_1,\dots,d_n\ge 0$ the matrix

$$\mathsf{J}=\begin{bmatrix} d_1 & -c_1 & & & \\ -c_1 & d_2 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{n-2} & d_{n-2} & -c_{n-1}\\ & & & -c_{n-1} & d_n\end{bmatrix}$$

is a singular M-matrix iff

$$d_1=\frac{c_1\omega_2}{\omega_1},\ d_j=\frac{1}{\omega_j}(c_j\omega_{j+1}+c_{j-1}\omega_{j-1}),\ d_n=\frac{c_{n-1}\omega_{n-1}}{\omega_n}$$

where ω is a weigth: $\omega_i>0$ and $\omega_1^2+\dots+\omega_n^2=1$

The Moore-Penrose inverse: $J^\dagger=(g_{ij})$

$$g_{ji}=g_{ij}=\omega_i\omega_j\left[\sum_{k=1}^{i-1}\frac{W_k^2}{c_k\omega_k\omega_{k+1}}+\sum_{k=j}^{n-1}\frac{(1-W_k)^2}{c_k\omega_k\omega_{k+1}}-\sum_{k=i}^{j-1}\frac{W_k(1-W_k)}{c_k\omega_k\omega_{k+1}}\right],\ i\le j,$$

where $W_k=\sum_{l=1}^{k}\omega_l^2$

- $g_{ij}<g_{ii+1},\ j=i+2,\dots,n$

When the Moore-Penrose inverse is an M–matrix?

- $g_{ii+1}\le 0$ for any $i=1,\dots,n-1$, or equivalently

$$\underbrace{\begin{bmatrix} \frac{W_1(1-W_1)}{\omega_1\omega_2} & -\frac{(1-W_2)^2}{\omega_2\omega_3} & -\frac{(1-W_3)^2}{\omega_3\omega_4} & \cdots & -\frac{(1-W_{n-1})^2}{\omega_{n-1}\omega_n}\\ -\frac{W_1^2}{\omega_1\omega_2} & \frac{W_2(1-W_2)}{\omega_2\omega_3} & -\frac{(1-W_3)^2}{\omega_3\omega_4} & \cdots & -\frac{(1-W_{n-1})^2}{\omega_{n-1}\omega_n}\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ -\frac{W_1^2}{\omega_1\omega_2} & -\frac{W_2^2}{\omega_2\omega_3} & \cdots & \frac{W_{n-2}(1-W_{n-2})}{\omega_{n-2}\omega_{n-1}} & -\frac{(1-W_{n-1})^2}{\omega_{n-1}\omega_n}\\ -\frac{W_1^2}{\omega_1\omega_2} & -\frac{W_2^2}{\omega_2\omega_3} & \cdots & -\frac{W_{n-2}^2}{\omega_{n-2}\omega_{n-1}} & \frac{W_{n-1}(1-W_{n-1})}{\omega_{n-1}\omega_n}\end{bmatrix}}_{\mathsf{A}(\omega)}\underbrace{\begin{bmatrix}\frac{1}{c_1}\\ \frac{1}{c_2}\\ \vdots\\ \frac{1}{c_{n-1}}\\ \frac{1}{c_n}\end{bmatrix}}_{\mathsf{c}^{-1}}\ge 0$$

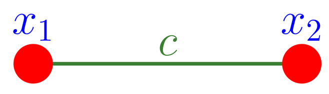
- If $\mathsf{A}(\omega)\mathsf{c}^{-1}\ge 0$, then $\mathsf{A}(\omega)$ is an M–matrix
- If $\mathsf{A}(\omega)$ is an invertible M–matrix, then $\mathsf{c}^{-1}=\mathsf{A}(\omega)\mathsf{a}$, where $\mathsf{a}\ge 0$
- If $\mathsf{A}(\omega)$ is a singular M–matrix, then $\mathsf{A}(\omega)\mathsf{c}^{-1}\ge 0$ iff $\mathsf{A}(\omega)\mathsf{c}^{-1}=0$, iff

$\mathsf{c}=\mathsf{t}\mathsf{c}(\omega)$, where $t>0$ and $c_j(\omega)=\frac{(1-W_j)}{\omega_j\omega_{j+1}}\prod_{k=j}^{n-2}\frac{W_k}{(1-W_k)},\ \ j=1,\dots,n-1$

Combinatorial Laplacian case: ω is constant

- Necessarily $n\le 4$ and $\frac{1}{2}\le\frac{c_1}{c_2}\le 2$ if $n=3$ or $c_1=c_3,\ c_2=2c_1$ if $n=4$

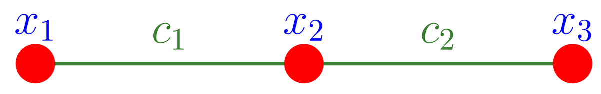
Low dimensions: $n=2$



For any $0<x<1$, if $\omega=\left(x,\sqrt{1-x^2}\right)$

$$\mathsf{J}=\begin{bmatrix}\frac{c\sqrt{1-x^2}}{x} & -c\\ -c & \frac{xc}{\sqrt{1-x^2}}\end{bmatrix}\quad\text{and}\quad \mathsf{J}^\dagger=\frac{x(1-x^2)}{c}\begin{bmatrix}\sqrt{1-x^2} & -x\\ -x & \frac{x^2}{\sqrt{1-x^2}}\end{bmatrix}$$

Low dimensions: $n=3$



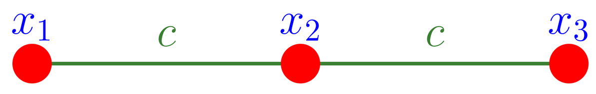
$\mathsf{J}^\dagger$ is an M–matrix iff $\frac{\omega_1^3}{\omega_3(1-\omega_3^2)}\le\frac{c_1}{c_2}\le\frac{\omega_1(1-\omega_1^2)}{\omega_3^3}$

- Given $c_1,c_2>0$, $\mathsf{J}^\dagger$ is an M–matrix for

$$\left\{\left(\omega_1,\sqrt{1-(1+t^2)\omega_1^2},t\omega_1\right):0<t<\frac{c_2}{c_1},\ 0<\omega_1\le\sqrt{\frac{tc_1}{c_2+t^3c_1}}\right\}$$

$$\cup\left\{\left(\omega_1,\sqrt{1-(1+t^2)\omega_1^2},t\omega_1\right):\frac{c_2}{c_1}\le t,\ 0<\omega_1<\sqrt{\frac{1}{1+t^2}}\right\}$$

- Example:



$t=1$ and $0<x<\frac{1}{2}$

$$\mathsf{J}=\begin{bmatrix}\frac{c\sqrt{1-2x^2}}{x} & -c & 0\\ -c & \frac{2xc}{\sqrt{1-2x^2}} & -c\\ 0 & -c & \frac{c\sqrt{1-2x^2}}{x}\end{bmatrix},\ \mathsf{J}^\dagger=\begin{bmatrix}\frac{x(1-2x^2+2x^4)}{c\sqrt{1-2x^2}} & -\frac{x^2(1-2x^2)}{c} & -\frac{2x^3(1-x^2)}{c\sqrt{1-2x^2}}\\ -\frac{x^2(1-2x^2)}{c} & \frac{2x^3\sqrt{1-2x^2}}{c} & -\frac{x^2(1-2x^2)}{c}\\ -\frac{2x^3(1-x^2)}{c\sqrt{1-2x^2}} & -\frac{x^2(1-2x^2)}{c} & \frac{x(1-2x^2+2x^4)}{c\sqrt{1-2x^2}}\end{bmatrix}$$

An example for $n=4$ and $\det(\mathsf{A})(\omega)>0$

$$\omega=\frac{1}{\sqrt{3(3+\sqrt5)}}\left(1,\frac{3+\sqrt5}{2},\frac{3+\sqrt5}{2},1\right)$$

$$\mathsf{J}=\begin{bmatrix}\left(\frac{3+\sqrt5}{2}\right)c & -c & 0 & 0\\ -c & \left(\frac{9-\sqrt5}{2}\right)c & -3c & 0\\ 0 & -3c & \left(\frac{9-\sqrt5}{2}\right)c & -c\\ 0 & 0 & -c & \left(\frac{3+\sqrt5}{2}\right)c\end{bmatrix}$$

$$\mathsf{J}^\dagger=\frac{1}{36(47+21\sqrt5)c}\begin{bmatrix}591+263\sqrt5 & -2(20+9\sqrt5) & -2(74+33\sqrt5) & -(99+43\sqrt5)\\ -2(20+9\sqrt5) & 177+79\sqrt5 & -(105+47\sqrt5) & -2(74+33\sqrt5)\\ -2(74+33\sqrt5) & -(105+47\sqrt5) & 177+79\sqrt5 & -2(20+9\sqrt5)\\ -(99+43\sqrt5) & -2(74+33\sqrt5) & -2(20+9\sqrt5) & 591+263\sqrt5\end{bmatrix}$$

An example for $n=4$ and $\det(\mathsf{A})(\omega)=0$

$$\omega=\frac{1}{6}\left(2,3\mp\sqrt5,3\pm\sqrt5,2\right)$$

$$\mathsf{J}=\begin{bmatrix}\left(\frac{3\mp\sqrt5}{2}\right)c & -c & 0 & 0\\ -c & (12\pm5\sqrt5)c & -3c & 0\\ 0 & -3c & (12\mp5\sqrt5)c & -c\\ 0 & 0 & -c & \left(\frac{3\pm\sqrt5}{2}\right)c\end{bmatrix}$$

$$\mathsf{J}^\dagger=\frac{1}{36c}\begin{bmatrix}16(3\pm\sqrt5) & 0 & -(14\pm3\sqrt5) & -12\\ 0 & 2(3\mp\sqrt5) & 0 & -(14\mp3\sqrt5)\\ -(14\pm3\sqrt5) & 0 & 2(3\pm\sqrt5) & 0\\ -12 & -(14\mp3\sqrt5) & 0 & 16(3\mp\sqrt5)\end{bmatrix}$$

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